Engineering Notes

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Optimization of Aircraft Altitude and Flight-Path Angle Dynamics

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Introduction

THIS Note addresses the optimization of aircraft altitude and flight-path angle dynamics in a form suitable for on-line computation and control. The analytic approach is a direct extension of a previous study,[1] where singular perturbation methods were used to optimize position, energy, and heading dynamics, and thus represent an optimal control solution that models all of the primary trajectory-related dynamics. The resulting algorithm can be regarded as a nonlinear feedback control law. Minimum time intercept of a fixed terminal point is used to set the framework in which the analytical results are developed.

The accuracy of the control solution depends on the degree to which altitude (h) and flight-path (γ) dynamics are separable in closed loop. Ardema[2,3] has shown, based on linearized eigenvalue analysis, that these dynamics are highly coupled when optimized on the same time scale. It is shown in this Note that the introduction of a penalty term on γ in the performance index improves the feedback solution obtained when these dynamics are optimized on separate time scales.

Problem Definition

To simplify the discussion, we consider the case of motion in the vertical plane, the simplified point mass dynamics, for which can be expressed as:

$$\dot{x} = V\cos\gamma \tag{1}$$

$$\dot{E} = (T-D)V/W \tag{2}$$

$$\dot{h} = V\sin\gamma \tag{3}$$

$$\dot{\gamma} = (L - W\cos\gamma)/mV \tag{4}$$

The dynamics are written in a coordinate frame fixed to a flat, nonrotating Earth. We further assume that W is constant and thrust acts along the velocity vector. The performance index is taken as

$$J = \int (1 + k\gamma^2/2)\,\mathrm{d}t \qquad k \geq 0 \tag{5}$$

where $k=0$ for the minimum time flight. The analysis is further simplified by ignoring aerodynamic limits. Thrust is assumed to be bounded by

$$T_{\min}(h,V) \leq T \leq T_{\max}(h,V) \tag{6}$$

and drag is modeled using

$$D = qsC_D = qsC_{D_0} + KL^2/qs \qquad q = \rho V^2/2 \tag{7}$$

The optimal control problem is: Given the initial state, and a specified final value for x, find L and T that minimize Eq. (5) subject to the constraint in Eq. (6).

For purposes of real-time control, we desire an approximate feedback solution; thus, we are led to the use of singular perturbation methods. In problems of this type, it is customary to regard position x as the slowest variable and energy E as faster than x, but slower than h and γ dynamics. Thus a perturbation parameter is artifically introduced into the dynamics:

$$\dot{x} = V\cos\gamma \tag{8}$$

$$\epsilon\dot{E} = (T-D)V/W \tag{9}$$

$$\epsilon^2\dot{h} = V\sin\gamma \tag{10}$$

$$\epsilon^2\dot{\gamma} = (L - W\cos\gamma)/mV \tag{11}$$

and an approximate solution is found by an asymptotic expansion of the state equations and necessary conditions about $\epsilon=0$, and enforcing all the boundary conditions. Since the expansion is nonuniform at $t=0$ and at $t=t_f$, boundary-layer solutions are required to satisfy the end conditions. This is accomplished by replacing t by the stretched time variables

$$\tau_i = t/\epsilon^i \qquad i = 1,2 \tag{12}$$

in the ith boundary layer. The outer and first (energy climb) boundary-layer solutions are summarized below.[1]

Outer solution:

$$h_0, E_0 = \arg\{\max_{h,E}(V)\} \tag{13}$$

$$\lambda_{x_0} = -1/V_0, \quad V_0 = \sqrt{(E_0 - h_0)2g}, \quad L_0 = W,$$
$$\gamma_0 = 0, \quad T_{\max} = D_0 \tag{14}$$

First boundary layer:

$$h_c^1 = \arg\left\{\max_h\left[\frac{(T_{\max} - D_0)V}{V_0 - V}\right]\right\}_{\substack{E=E_{\text{current}}\\ T_{\max}>D_0}} \tag{15}$$

$$\lambda_{E_1} = -WH_0(E,h_1)/V_1(T_{\max} - D_0),$$
$$V_1 = \sqrt{(E - h_1)2g}, \quad L_1 = W \tag{16}$$

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where

$$H_0 = \lambda_{x_0} V_1 + 1 \tag{17}$$

$$D_0 = q_1 s C_{D_0} + KW^2/q_1 s, \qquad q_1 = \rho(h_1) v_1^2/2 \tag{18}$$

For this problem Eqs. (13)-(16) can be precomputed and stored as a function of E. A second boundary-layer solution is needed to match the boundary conditions on h and γ, using L and T as control variables. However, this leads to a nonlinear two-point boundary value problem. Thus we are led to consider a further separation of h and γ dynamics.

$$\epsilon^2 \dot{h} = V\sin\gamma \tag{19}$$

$$\epsilon^3 \dot{\gamma} = (L - W\cos\gamma)/mV \tag{20}$$

This leads to the following closed-loop solution during climb:[4]

Second Boundary Layer:

$$\gamma_2 = \arg\left\{\max_{\gamma}\left[\frac{\sin\gamma}{H_1(E,h,\gamma)}\right]\right\}\mathrm{sign}(h_1^c - h) \tag{21}$$

$$\lambda_{h_2} = -H_1/V\sin\gamma_2, \qquad L_2 = W \tag{22}$$

where

$$H_1 = \lambda_{x_0} V\cos\gamma + \lambda_{E_1}(T_{\max} - D_1)V/W + 1 + k\sin^2\gamma/2 \tag{23}$$

$$D_1 = qsC_{D_0} + KW^2\cos^2\gamma/qs \tag{24}$$

Third boundary layer:

$$\Delta L = \sqrt{-qsWH_2(E,h,\gamma)/\lambda_{E_1} VK}\ \mathrm{sign}(\gamma_2 - \gamma) \tag{25}$$

$$L_3 = W\cos\gamma + \Delta L \tag{26}$$

where

$$H_2 = H_1 + \lambda_{h_2} V\sin\gamma \tag{27}$$

The accuracy of the above control solution depends on the degree to which the h and γ dynamics are separable in closed loop. The purpose of this Note is to examine the extent to which the above control solution, when implemented for the combined h, γ system of dynamics, approximates the second boundary-layer solution dynamics for the formulation in Eqs. (8)-(11).

Eigenvalue Analysis

We first consider an expansion of the second boundary layer necessary conditions for the formulation in Eqs. (8)-(11), for $k=0$:

$$\mathrm{d}h/\mathrm{d}\tau_2 = V\sin\gamma \qquad \mathrm{d}\lambda_h/\mathrm{d}\tau_2 = -\partial H_2/\partial h \tag{28}$$

$$\mathrm{d}\gamma/\mathrm{d}\tau_2 = (L - W\cos\gamma)/mV \qquad \mathrm{d}\lambda_\gamma/\mathrm{d}\tau_2 = -\partial H_2/\partial\gamma \tag{29}$$

$$H_2 = H_1 + \lambda_h V\sin\gamma + \lambda_\gamma(L - W\cos\gamma)/mV \tag{30}$$

$$\partial H_2/\partial L = 0 \tag{31}$$

Expanding Eqs. (28), (29), and (31) about the equilibrium conditions,

$$\bar{h} = h_1^c(E), \qquad \bar{\gamma} = 0 \tag{32}$$

$$\bar{\lambda}_h = 0, \qquad \bar{\lambda}_\gamma = 4m\lambda_{E_1}K/\bar{\rho}s < 0 \tag{33}$$

$$\bar{L} = W \tag{34}$$

where the value for $\bar{\lambda}_\gamma$ follows from Eq. (31) for $\bar{L} = W$. Substituting for L from Eq. (31) results in the linear perturbation equations:

$$\frac{\mathrm{d}}{\mathrm{d}\tau_2}\begin{bmatrix}\delta h\\ \gamma\\ \lambda_h\\ \delta\lambda_\gamma\end{bmatrix} = \begin{bmatrix}0 & V_1 & 0 & 0\\ g\bar{\rho}_h/V_1\bar{\rho} & 0 & 0 & g/V_1\bar{\lambda}_\gamma\\ K_1 & 0 & 0 & -g\bar{\rho}_h/V_1\bar{\rho}\\ 0 & K_2 & -V_1 & 0\end{bmatrix} \times \begin{bmatrix}\delta h\\ \gamma\\ \lambda_h\\ \delta\lambda_\gamma\end{bmatrix} \tag{35}$$

where

$$\bar{\rho}_h = \partial\rho/\partial h\,|_{\bar{h}} \tag{36}$$

$$K_1 = -\partial^2 H_1/\partial h^2 \leq 0 \tag{37}$$

$$K_2 = \lambda_{x_0} V_1 - \bar{\lambda}_\gamma g/V_1 \tag{38}$$

The dependancy of C_{D_0} and K on Mach number is ignored in Eq. (37). The resulting eigenvalues are the roots of

$$s^4 + as^2 + b = 0 \tag{39}$$

where

$$a = g(g/V_1^2 - 2\bar{\rho}_h/\bar{\rho}) - \lambda_{x_0} g/\bar{\lambda}_\gamma \tag{40}$$

$$b = gV_1K_1/\bar{\lambda}_\gamma \geq 0 \tag{41}$$

and are arranged symmetrically about the real and imaginary axes. A necessary condition for a stable boundary-layer solution is that none of the eigenvalues lie on the imaginary axis.[2]

We next consider the linearized boundary-layer dynamics for the same system using the control solution in Eq. (21)-(27). The modeling of relative position dynamics appears to

Table 1 Comparison of eigenvalues for h, γ boundary-layer dynamics

Energy level, m	Eq. (39)	Eigenvalues (1/s), Eq. (47), $k=0$	Eq. (47), $k=1.78$
9112	$-0.156 \pm i0.108$	$-0.082 \pm i0.171$	$-0.156 \pm i0.108$
11320	$-0.114 \pm i0.078$	$-0.064 \pm i0.123$	$-0.123 \pm i0.063$
13528	$-0.089 \pm i0.068$	$-0.048 \pm i0.100$	$-0.093 \pm i0.062$
15737	$-0.069 \pm i0.064$	$-0.032 \pm i0.089$	$-0.064 \pm i0.070$
17945	$-0.084 \pm i0.072$	$-0.038 \pm i0.105$	$-0.067 \pm i0.090$

be essential for the solution of Eq. (21) to be asymptotic since $\gamma_2 = \pm 90$ deg (zoom climb or dive) when $\lambda_{x_0} = 0$.[5] Whereas, in this case, $\gamma_2 \to 0$ as $(h_1^c - h) \to 0$, even for $k=0$. Figure 1 illustrates the dependence of γ_2 on $(h_1^c - h)$ and k for the F-8 aircraft.

Expanding the dynamics in Eqs. (28) and (29), we obtain

$$\mathrm{d}\delta h/\mathrm{d}\tau_2 = V_1 \gamma \tag{42}$$

$$\mathrm{d}\gamma/\mathrm{d}\tau_2 = \delta L/mV \tag{43}$$

It can be shown that δL is related to γ and h by

$$\delta L = -K_3\gamma - K_4 \delta h \tag{44}$$

where

$$K_3 = K_5\sqrt{\partial^2 H_2/\partial\gamma^2}, \qquad K_4 = K_5\sqrt{\partial^2 H_2^2/\partial h^2} \tag{45}$$

$$K_5 = \sqrt{-qsW/2\lambda_{E_1} K V_1} \tag{46}$$

Thus, the eigenvalues of the closed-loop system are given by the roots of

$$s^2 + K_3 s/mV_1 + K_4/m = 0 \tag{47}$$

It can be shown that the weighting parameter k enters into K_3 but not K_4. Thus, it affects the damping ratio but not the natural frequency in Eq. (47). Also note that k enters the control solution through Eq. (23).

Numerical Results

Aerodynamic and propulsion data for an F-8 aircraft was used to calculate and compare the eigenvalues from Eqs. (39) and (47). The calculation was performed at five energy levels along the climb path. For this aircraft, the long-range cruise energy is $E_0 = 19{,}100$ m. The first three columns in Table 1 compare the eigenvalues for $k=0$ (minimum time). Note that in every case the damping ration obtained using Eq. (47) is approximately half that obtained from Eq. (39). What is not apparent from Table 1 is that the natural frequencies are equal.

The weighting parameter k only affects the damping ratio and not the natural frequency of the closed-loop dynamics corresponding to Eq. (47). Thus, it is possible to tune the solution for a single energy level. For example, for $E = 9112$ m and $k = 1.78$, the eigenvalues from Eqs. (39) and (47) are equal. The calculation at other energy levels is summarized in the last column of Table 1. Note that a reasonably good approximation to the eigenvalues of Eq. (39) is obtained. A better approximation results when k is chosen for $E = 13{,}528$ m. In any case, the high degree of coupling that exists between h and γ dynamics is evidenced by the fact that the eigenvalues of Table 1 occur in complex conjugate pairs at all energy levels. The improved damping that results for $k>0$ has been verified through nonlinear simulation of the F-8 dynamics.[4]

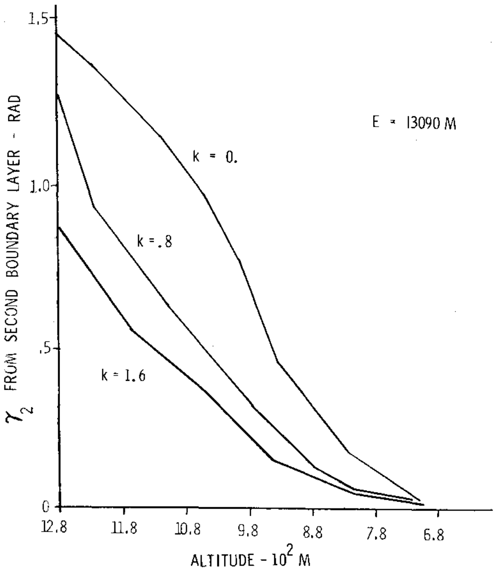

Fig. 1 Optimum flight-path angle from third boundary-layer solution.

Summary

This Note proposes a method for optimizing h and γ dynamics in a form suitable for on-line computation. It is shown that formal separation and boundary solution for these dynamics results in a control solution with insufficient damping. The eigenvalues of the linearized closed-loop dynamics along the optimal climb path can be made to closely approximate the linearized dynamics of the exact solution by introducing a penalty term on γ in the performance index.

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A Flexible Structure Controller Design Method Using Mode Residualization and Output Feedback

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Introduction

PROPOSED large flexible space systems (LFSS) will involve multiple actuators and sensors, providing the prospect of integrated control algorithms for attitude control and structural mode damping. The analysis and synthesis of

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